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ABSTRACT

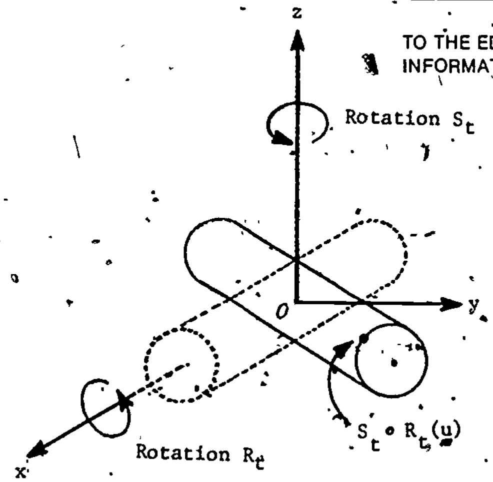
This module applies linear algebraic methods to solve the following problem: If an object in a three-dimensional coordinate system is first rotated about a given axis through the origin by a given angle, and then rotated about another axis through the origin by another angle, there is a straightforward way to calculate the combined result of the two rotations. It is felt that the method presented to answer the question is even more interesting than the question itself. The material includes exercises and a model exam. Answers are provided for both. (MP)

umap UNIT 313

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

USING THE QUATERNIONS TO COMPOSE ROTATIONS

by Frederick Solomon



APPLICATIONS OF LINEAR ALGEBRA TO GEOMETRY

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USING THE QUATERNIONS TO COMPOSE ROTATIONS

by

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Intermodular Description Sheet: UMAP Unit 313

Title: USING THE QUATERNIONS TO COMPOSE ROTATIONS

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Prerequisite Skills:

1. Understand Euclidean n-space as a vector space.
2. Understand polar representation of the complex numbers.
3. Understand the notion of linear operator.

Output Skills:

1. To calculate the result of two rotations about different axes in 3-space.
2. To learn the definition of the quaternions and some of its algebraic properties.

Other Related Units:

MODULES AND MONOGRAPHS IN UNDERGRADUATE

MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

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USING THE QUATERNIONS TO COMPOSE ROTATIONS

1. INTRODUCTION

This unit applies linear algebraic methods to solve an easily stated problem: Suppose an object in a three-dimensional coordinate system is first rotated about a given axis through the origin by a given angle, and then rotated about another axis through the origin by another angle. Is there a straightforward way to calculate the combined result of the two rotations? For example, are there formulas that describe the result in terms of the two axes of rotation and the two angles? It turns out that the answer is yes, as we shall see.

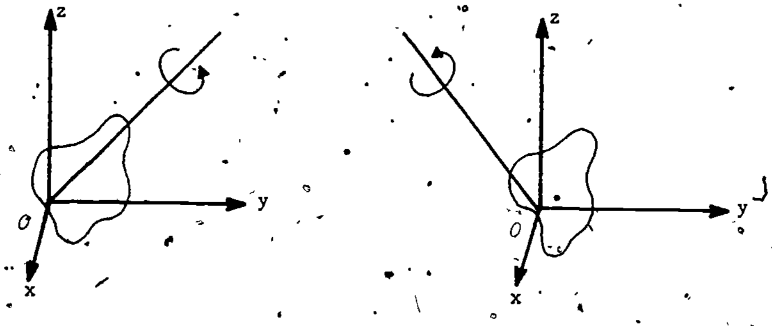


Figure 1. Rotations of an object about different axes through the origin.

Although this is an interesting question to answer, the method by which we will answer it is more interesting still. For, in considering rotations in Euclidean three-dimensional space R^3, we embed R^3 in the four-dimensional space R^4. In R^4 there is a vector product which generalizes the cross product in R^3 and which is intimately related to rotations in R^3.

We define a rotation in R^3 as a rotation about a fixed axis vector $\underline{n}$ by a given angle θ. The vector $\underline{n}$ is

assumed to emanate from the coordinate origin, and *we will hold to this assumption throughout the unit.* The sense of rotation is this: If the right-hand thumb points in the direction of $\underline{n}$, then the fingers curl in the direction of rotation; and only vectors perpendicular to $\underline{n}$ are rotated. We will take the axis of rotation $\underline{n}$ to be a *unit* vector. Notice that the rotation about $\underline{n}$ by angle θ is the same as the rotation about $-\underline{n}$ by angle $2\pi-\theta$.

A rotation is a *linear* operator on R^3. That is, if $R(\underline{u})$ denotes the vector obtained by rotating $\underline{u}$ about $\underline{n}$ by angle θ, then

$$R(\underline{u} + \underline{v}) = R(\underline{u}) + R(\underline{v}), \quad \text{and} \quad R(r\underline{u}) = rR(\underline{u}),$$

for any vectors $\underline{u}$ and $\underline{v}$ and scalar r. To see that $R(\underline{u} + \underline{v}) = R(\underline{u}) + R(\underline{v})$, for example, note that $\underline{u} + \underline{v}$ is a diagonal of the parallelogram determined by $\underline{u}$ and $\underline{v}$. After the rotation, the corresponding diagonal of the parallelogram determined by $R(\underline{u})$ and $R(\underline{v})$, namely $R(\underline{u}) + R(\underline{v})$, is exactly the diagonal of the original parallelogram rotated, $R(\underline{u} + \underline{v})$. See Figure 2.

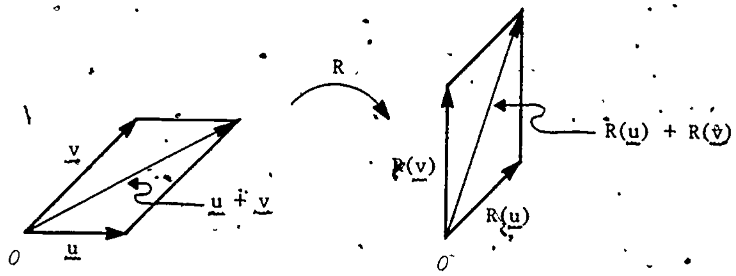


Figure 2. Under a rotation R in R^3, the image of the sum of two vectors $\underline{u}$ and $\underline{v}$ is the image of their sum.

Similarly, changing the length (and perhaps the direction) of a vector by multiplying the vector by a scalar r, whether performed before or after the rotation, yields the same result. Thus $R(r\underline{u}) = rR(\underline{u})$. Consequently, a rotation is a linear operator.

It is convenient to express rotations by using a "right-hand" set of orthonormal vectors in R^3. Let n be the unit vector defining the axis of rotation as above. Let u and v each be unit vectors so that u, v, n, are mutually perpendicular, with cross product $u \times v = n$. Then the rotation about n by angle θ is given by

$$R(n) = n$$

$$R(u) = \cos\theta\, u + \sin\theta\, v$$

$$R(v) = -\sin\theta\, u + \cos\theta\, v$$

in the basis u, v, n of R^3. See Figure 3.

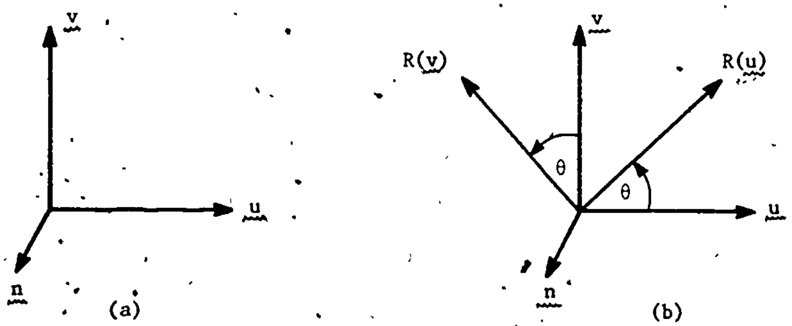


Figure 3. (a) Three mutually orthogonal vectors u, v and n, pictured with n pointing out of the page. (b) A rotation R about n by an angle θ (shown here as a small positive angle) moves u to $R(u)$ and v to $R(v)$.

2. CONTEXTS IN WHICH ROTATIONS ARE COMPOSED

We now indicate two contexts in which applications of compositions of rotations occur.

For the first context, consider two reference frames—an unprimed one with axes x, y, and z and a primed reference frame with axes x', y', and z'. Assume that each is a right-hand Cartesian frame. That is, in each the x, y, and z axes are mutually perpendicular with the positive z axis obtained by the right-hand rule from the x and y axes as in several variable calculus. Assume

that the axes are all marked off with the same units, and that the coordinate origins of the two systems coincide. Let i, j, k and i', j', k' be the unit vectors along the x, y, z axes, respectively, in each system. Then there exists a rotation R that takes i to i', j to j', and k to k'. That is,

$$i' = R(i) \qquad j' = R(j) \qquad k' = R(k).$$

To see that such a rotation exists let S be *any* rotation with $S(i) = i'$. Now rotate around the axis $S(i) = i'$ until $S(j)$ and $S(k)$ coincide respectively with j' and k'. Call this new rotation T. Then the composition of T and S yields

$$T \circ S(i) = i' \qquad T \circ S(j) = j' \qquad T \circ S(k) = k'.$$

See Figure 4.

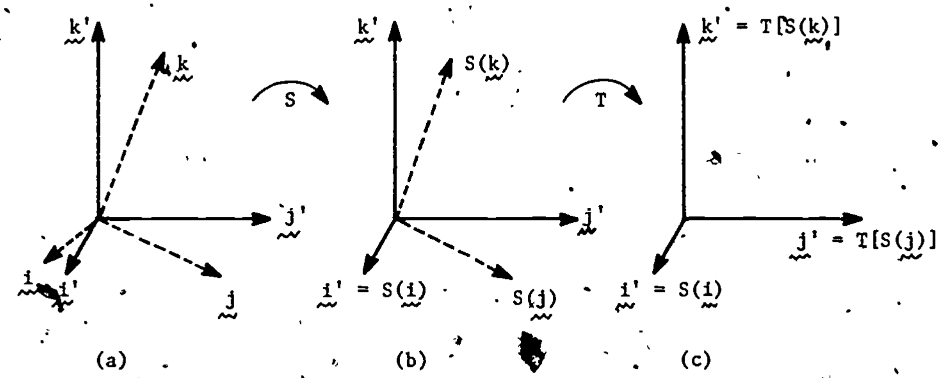


Figure 4. The rotation S about the origin takes i to i', as shown in parts (a) and (b). Then, the rotation T about the origin takes $S(j)$ to j' and $S(k)$ to k'.

Now, the composition of rotations about axes that pass through a common point is itself a rotation (a direct consequence of Theorem 3 of Section 10 below), so that $T \circ S$ is the desired rotation R. Thus we have the result that *given any two right-hand Cartesian coordinate systems whose origins coincide, there exists a rotation which maps the x, y, z axes in one system onto the corresponding*

axes in the second system: further, the positive sense of each axis is preserved.

Now consider three right-hand Cartesian-coordinate systems whose origins coincide. Call them unprimed, primed and doubly primed. Suppose we know a rotation taking the unprimed system to the primed system, and also a rotation taking the primed to the doubly primed. That is, let R_1 and R_2 be the rotations so that

$$\underset{\sim}{i}' = R_1(\underset{\sim}{i}), \quad \underset{\sim}{j}' = R_1(\underset{\sim}{j}), \quad \underset{\sim}{k}' = R_1(\underset{\sim}{k}),$$

$$\underset{\sim}{i}'' = R_2(\underset{\sim}{i}'), \quad \underset{\sim}{j}'' = R_2(\underset{\sim}{j}'), \quad \underset{\sim}{k}'' = R_2(\underset{\sim}{k}').$$

Then the transformation taking the unprimed axes to the doubly primed axes is the composition of the rotations:

$$\underset{\sim}{i}'' = R_2 \circ R_1(\underset{\sim}{i}), \quad \underset{\sim}{j}'' = R_2 \circ R_1(\underset{\sim}{j}), \quad \underset{\sim}{k}'' = R_2 \circ R_1(\underset{\sim}{k}).$$

As another context in which it would be useful to know how to compose rotations, consider an object which is rotating about two axes simultaneously. A springboard diver who is twisting and spinning at the same time exhibits such behavior. So does the orbiter shown in Figure 5. To be specific, consider a cylinder whose axis initially lies along the x-axis and whose center of mass lies at the origin (Figure 6). Suppose the cylinder is rotating about this axis so that at time t the total angle through which the cylinder has rotated is $\theta(t)$. (For "uniform" rotation, $\theta(t) = ct$ for some constant c.) Now suppose that the cylinder is also rotating about the z-axis so that at time t the total angle of rotation is $\phi(t)$. Let R_t denote the rotation about the x-axis by angle $\phi(t)$ and S_t denote the rotation about the z-axis by angle $\phi(t)$. Then a point on the cylinder whose position is $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ at time 0 will be at $S_t \circ R_t(\underset{\sim}{u})$ at time t. That is, the location of $\underset{\sim}{u}$ after t units of time have elapsed is found by first spinning about the x-axis and then about the z-axis even though in reality the two rotations occur simultaneously.

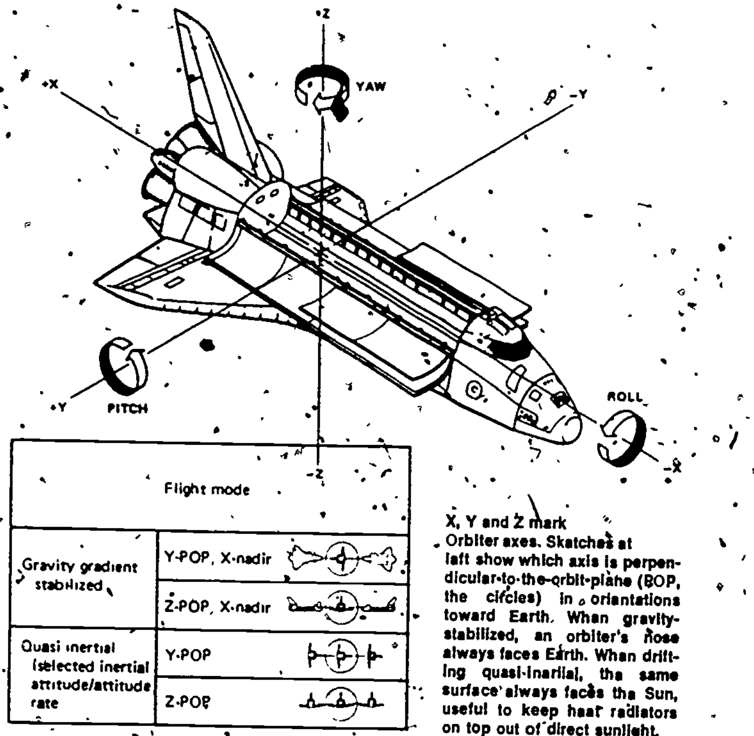
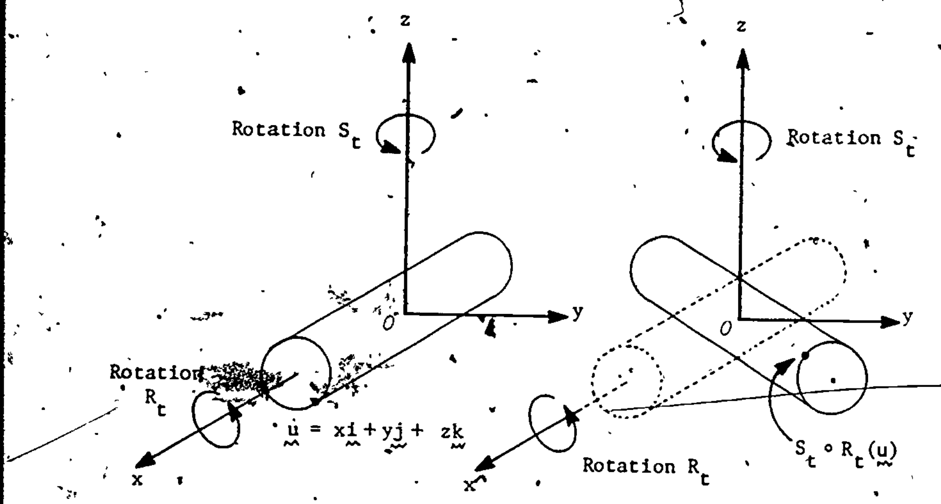


Figure 5. Orbiter.

(Source: *Air and Space*, Volume 2, No. 3, Jan.-Feb. 1979, page 6.)



(a) Position of u at time $t = 0$.

(b) Position of u after R and S have acted simultaneously for t units of time.

Figure 6. The point u is moved by the rotations R_t and S_t, as described in the text. Its position may be found by spinning first about the x-axis, and then about the z-axis.

3. THE QUATERNIONS

Four-dimensional Euclidean space R^4 can be given a product function; with this product R^4 is called *the quaternions*, and denoted by the letter Q. The essential feature of this product is that it permits a kind of division, whereas the cross product in R^3 does not. Let us begin by denoting an element (a,b,c,d) of R^4 by $a + bi + cj + dk$, just as in R^3 the element (b,c,d) is often denoted by $bi + cj + dk$. In this way we may *embed* R^3 in $R^4 = Q$ as the last three coordinates. In particular we may identify the vectors on the left side of the following table with the symbols on the right:

$(1,0,0,0)$ with 1

$(0,1,0,0)$ with i

$(0,0,1,0)$ with j

$(0,0,0,1)$ with k.

We will denote scalars by lowercase letters a, b, c, and d. Vectors either in R^3 or R^4 will be denoted by the lowercase letters p, q, u, v, i, j, and k. A vector of the form $u = ai + bj + ck$ can be thought to be *either* in R^3 or in R^4 (as $u = 0 + ai + bj + ck$). However, in performing the cross product $u \times v$, both u and v must be in R^3. A quaternion of the form $(a,0,0,0) = a1$ will often be denoted just by a.

4. ADDITION AND MULTIPLICATION

Addition is defined componentwise just as in linear algebra. To define the multiplication, we give the rules for multiplying the basis vectors 1, i, j, k and then "extending by linearity." These products are

$$i^2 = ii = -1, \qquad ij = k = -ji,$$

$$j^2 = -1, \qquad ki = j = -ik,$$

$$k^2 = -1, \qquad jk = i = -kj.$$

The vector $1 = (1,0,0,0)$ is to behave as a multiplicative unit. That is,

$$1\,q = q = q\,1,$$

for any quaternion q.

Example: To find the product of $2+3i$ and $-1+6k$, we write

$$(2 + 3i)(-1 + 6k) = 2(-1) + 2(6k) + 3i(-1) + 3i(6k)$$

$$= -2 + 12k - 3i + 18ik$$

$$= -2 - 3i - 18j + 12k.$$

Exercise 1: Show that the complex numbers can be embedded in Q by carrying out the following steps. Consider any two-dimensional plane containing the $1 = (1,0,0,0)$ axis. That is, let P be the set of vectors of the form $a + bn$ where n is a fixed unit vector of the form $x_1 i + x_2 j + x_3 k$ and where a and b are free to take on any scalar (real number) values. (A vector $u = x_0 + x_1 i + x_2 j + x_3 k$ is a unit vector if $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$.) Show that multiplication of quaternions in P has the usual rules for the multiplication of complex numbers. In particular show that $n^2 = -1$.

Exercise 2: Let $u = ai + bj + ck$ and $v = xi + yj + zk$. Show that multiplication of u and v as quaternions yields

$$uv = -(u \cdot v) + (u \times v)$$

where $\cdot$ indicates the dot or inner product and $\times$ denotes the usual cross product in R^3. Conclude that the following two formulas hold for vectors u and v in R^3:

(1) $u \cdot v = -\frac{1}{2}(uv + vu)$,

(2) $u \times v = \frac{1}{2}(uv - vu)$,

where the products on the right denote quaternionic multiplication.

5. CONJUGATION AND NORM

Two other operations frequently encountered in work with quaternions are conjugation and norm. Let $q = a + bi + cj + dk$. We define the *conjugate* of q to be

$$\bar{q} = a - bi - cj - dk.$$

The *norm* of q is defined to be the usual length of q in R^4:

$$|q| = \sqrt{(a^2 + b^2 + c^2 + d^2)}.$$

Exercise 3: Show that the following properties hold for any quaternions p and q:

(1) $q\bar{q} = |q|^2 = \bar{q}q$.

(2) $\bar{\bar{q}} = q$.

(3) $\overline{pq} = \bar{q}\,\bar{p}$

(4) $|pq| = |p||q|$.

Notice that the third property says that the conjugate of a product is the product of the conjugates *in the reverse order*. This reversal of order holds for products of more than two quaternions as well. Notice also that the first property implies that every $q \neq 0$ has a *multiplicative inverse*. For any $q \neq 0$, there is a p so that $pq = 1 = qp$; namely, $p = \bar{q} / |q|^2$.

6. THE QUATERNIONS ARE A SKEW-FIELD

All the "normal" rules of arithmetic hold for multiplication and addition in Q *except* commutativity of multiplication. For example, $ij = k \neq -k = ji$. Here are the basic properties of multiplication, those for addition being the ones with which you are already familiar from linear algebra:

Associativity: $(pq)r = p(qr)$.

Distributivity: $p(q+r) = pq + pr$

$(q+r)p = qp + rp$.

Multiplicative Identity: The quaternion 1 satisfies

$$1p = p = p1$$

for all quaternions p.

Reciprocals: For any $q \neq 0$ there is a p so that

$$qp = 1 = pq.$$

Although associativity and distributivity are somewhat tedious to prove, their proofs follow quite directly from

the definition of multiplication. The existence of reciprocals allows division although there is ambiguity. For example, to divide q_1 by q_2 we may multiply q_1 by the reciprocal of q_2 either on the left or the right. And these may not yield the same result since multiplication is not in general commutative. With these properties the quaternions Q are called a *skew-field*.

If we think of Q as the vector space R^4, then for every real scalar r we have

$$r(a + bi + cj + dk) = ra + rbi + rcj + rdk.$$

However, we may think of the scalar r as $r1 = (r,0,0,0)$, in which case

$$\begin{aligned} r(a + bi + cj + dk) &= ra + rb1i + rc1j + rd1k \\ &= ra + rbi + rcj + rdk \end{aligned}$$

by the definition of multiplication in Q. Thus we arrive at the same result whether we think of scalars as real numbers or as quaternions with second, third and fourth coordinates zero.

7. POLAR REPRESENTATION OF QUATERNIONS

For many purposes, when dealing with complex numbers, the polar representation simplifies notation and calculation. Thus we denote a complex number $a+b\sqrt{-1}$ by $r(\cos\theta + \sin\theta\sqrt{-1})$ where

$$r = \sqrt{(a^2 + b^2)}, \quad \cos\theta = a/r, \quad \sin\theta = b/r.$$

We can develop an analogous representation for quaternions. To see this let $q = a + bi + cj + dk$, and let n be the unit vector in R^3 in the same direction as $bi + cj + dk$. This means that

$$n = \frac{bi + cj + dk}{\sqrt{b^2 + c^2 + d^2}}.$$

Now write

$$\begin{aligned} q &= a + bi + cj + dk \\ &= a + \sqrt{b^2 + c^2 + d^2}\, n \\ &= r\left[\frac{a}{r} + \frac{\sqrt{b^2 + c^2 + d^2}}{r} n\right], \end{aligned}$$

where $r = \sqrt{a^2 + b^2 + c^2 + d^2} = |q|$.

Since $\left(\frac{a}{r}\right)^2 + \left(\frac{\sqrt{b^2 + c^2 + d^2}}{r}\right)^2 = 1,$

the point $(a/r, \sqrt{(b^2 + c^2 + d^2)}/r)$ lies on the unit circle in R^2. Thus its coordinates are $(\cos\theta, \sin\theta)$ for some angle θ. We have now arrived at a *polar representation* for quaternions. Any quaternion q can be written in the form

$$q = |q|(\cos\theta + \sin\theta\, n)$$

for some angle θ and some unit vector n in R^3.

Exercise 4: Find the polar representations of the following quaternions.

(a) $i + j + k$

(b) $1 + i + j + k$

(c) $1 + k$

(d) $3 + 3j - 6k$.

---

8. DEFINITION OF THE MULTIPLICATION LINEAR OPERATOR - THE BASIC THEOREM

We now define a certain linear operator on Q. Since the space R^4 that underlies Q is a vector space, the notion of a linear operator on Q makes sense. It will turn out that this operator is intimately related to rotations in R^3. This is the basis for our gaining insight into rotations by considering Q. To define this

operator let q be a *fixed* quaternion. Then define the function $M_q: Q \to Q$ by the rule

$$M_q(p) = qp\bar{q}.$$

That is, the image of p is obtained by multiplying p on the left by q and on the right by $\bar{q}$. We will call the function M_q a *multiplication map*.

THEOREM 1: (a) The multiplication map M_q is a linear operator on Q; for any quaternion q. (b) If q_1 and q_2 are two quaternions, the composite of M_{q_2} followed by M_{q_1} is

$$M_{q_1} \circ M_{q_2} = M_{q_1 q_2}$$

(c) If $u = ai + bj + ck$ is in R^3 (has first coordinate zero) then its image $M_q(u)$ is also in R^3. Hence M_q, when restricted to R^3, has its range in R^3.

Proof of (a):

To show that M_q is linear, we must show for any two quaternions p_1 and p_2 and any scalar r that

$$M_q(p_1 + p_2) = M_q(p_1) + M_q(p_2)$$

and

$$M_q(rp_1) = rM_q(p_1).$$

But

$$\begin{aligned} M_q(p_1 + p_2) &= q(p_1 + p_2)\bar{q} \\ &= q(p_1\bar{q} + p_2\bar{q}) \\ &= qp_1\bar{q} + qp_2\bar{q} \\ &= M_q(p_1) + M_q(p_2), \end{aligned}$$

by the distributivity of multiplication. Similarly, M_q preserves scalar multiplication.

Proof of (b):

We show that $M_{q_1} \circ M_{q_2}(p) = M_{q_1 q_2}(p)$ for any quaternion p. This is accomplished by the following sequence of equalities:

$$\begin{aligned} M_{q_1} \circ M_{q_2}(p) &= M_{q_1}(q_2 p\bar{q_2}) \\ &= q_1(q_2 p\bar{q_2})\bar{q_1} \\ &= (q_1q_2)p(\bar{q_2}\bar{q_1}) && \text{by associativity} \\ &= (q_1q_2)p\overline{(q_1q_2)} && \text{by Exercise 3(3)} \\ &= M_{q_1q_2}(p). \end{aligned}$$

Proof of (c):

A quaternion u has first coordinate zero (is in R^3) if and only if $\bar{u} = -u$. We may therefore establish part (c) of the theorem by showing that if $\bar{u} = -u$, then $\overline{M_q(u)} = -M_q(u)$. The argument is brief:

$$\begin{aligned} \overline{M_q(u)} &= \overline{qu\bar{q}} \\ &= \bar{\bar{q}}\,\bar{u}\,\bar{q} && \text{by Exercise 3} \\ &= -qu\bar{q} && \text{if } \bar{u} = -u \\ &= -M_q(u). \end{aligned}$$

9. THE MULTIPLICATION OPERATOR IS "REALLY" A ROTATION

We may now develop the connection between the multiplication operator M_q and rotations in R^3. Let $n = ai + bj + ck$ be a vector of unit length in R^3. So for any angle θ, the quaternion $\cos\theta + \sin\theta\, n$ is a quaternion of unit length.

THEOREM 2: The multiplication map $M_{\cos\theta + \sin\theta \mathbf{n}}$, as a map from R^3 to R^3, is the rotation about axis $\mathbf{n}$ of angle 2θ. That is, if $\mathbf{u}$ is in R^3 then

$$M_{\cos\theta + \sin\theta \mathbf{n}}(\mathbf{u})$$

is obtained by rotating $\mathbf{u}$ about $\mathbf{n}$ by angle 2θ.

Proof:

We know from Theorem 1 that $M = M_{\cos\theta + \sin\theta\mathbf{n}}$ is a linear map from R^3 to R^3. To check that it is the required rotation we need only check its action on a basis for R^3. As in the introduction, we use a right-hand orthonormal basis $\mathbf{u},\mathbf{v},\mathbf{n}$.

There are three things to show, namely that

$$M(\mathbf{u}) = \cos(2\theta)\,\mathbf{u} + \sin(2\theta)\,\mathbf{v},$$

$$M(\mathbf{v}) = -\sin(2\theta)\,\mathbf{u} + \cos(2\theta)\,\mathbf{v},$$

and

$$M(\mathbf{n}) = \mathbf{n}.$$

We will establish the first of the three equations, and leave the verification of the remaining two to be done as Exercise 5. Here is the argument that $M(\mathbf{u})$ is what it ought to be. First of all.

$$\begin{aligned} M(\mathbf{u}) &= (\cos\theta + \sin\theta\,\mathbf{n})\,\mathbf{u}\,(\cos\theta - \sin\theta\,\mathbf{n}) \\ &= (\cos\theta\,\mathbf{u} + \sin\theta\,\mathbf{nu})(\cos\theta - \sin\theta\,\mathbf{n}) \\ &= \cos^2\theta\,\mathbf{u} - \sin^2\theta\,\mathbf{nun} + \sin\theta\cos\theta(\mathbf{nu} - \mathbf{un}). \end{aligned}$$

But $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{n}$ form a right-hand or orthonormal set. So

$$\mathbf{nu} = -(\mathbf{n}\cdot\mathbf{u}) + (\mathbf{n}\times\mathbf{u}) = 0 + \mathbf{v};$$

and

$$\mathbf{nun} = (\mathbf{nu})\mathbf{n} = \mathbf{vn} = -(\mathbf{v}\cdot\mathbf{n}) + (\mathbf{v}\times\mathbf{n}) = \mathbf{u}.$$

By Formula (2) in Exercise 2,

$$\mathbf{nu} - \mathbf{un} = 2\mathbf{n}\times\mathbf{u} = 2\mathbf{v}.$$

Therefore,

$$\begin{aligned} M(\mathbf{u}) &= (\cos^2\theta - \sin^2\theta)\mathbf{u} + 2\sin\theta\cos\theta\,\mathbf{v} \\ &= \cos(2\theta)\,\mathbf{u} + \sin(2\theta)\,\mathbf{v}, \end{aligned}$$

by the trigonometric formulas for the sine and cosine of twice an angle.

Exercise 5: Show that

$$M(\mathbf{v}) = -\sin(2\theta)\,\mathbf{u} + \cos(2\theta)\,\mathbf{v}$$

and $\quad M(\mathbf{n}) = \mathbf{n},$

thus completing the proof of Theorem 2.

---

10. APPLICATIONS TO ROTATIONS

We can now recast some of the results that we have obtained about the relation of the quaternion multiplication maps to rotations in R^3 in the form of a theorem, for future reference.

THEOREM 3: Let R_1 be a rotation about a unit vector $\mathbf{n}_1$ in R^3 by angle θ_1, and let R_2 be a rotation about a unit vector $\mathbf{n}_2$ by angle θ_2. Then

$$R_1 = M_{\mathbf{q}_1}, \quad \text{where } \mathbf{q}_1 = \cos\frac{\theta_1}{2} + \sin\frac{\theta_1}{2}\mathbf{n}_1,$$

and

$$R_2 = M_{\mathbf{q}_2}, \quad \text{where } \mathbf{q}_2 = \cos\frac{\theta_2}{2} + \sin\frac{\theta_2}{2}\mathbf{n}_2.$$

Then the composition of the rotation R_1 followed by R_2 is given by

$$R_2 \circ R_1 = M_{\mathbf{q}_2} \circ M_{\mathbf{q}_1} = M_{\mathbf{q}_2\mathbf{q}_1}.$$

In particular, the composition is itself a rotation, and may be achieved by multiplication by the quaternion $\mathbf{q}_2\mathbf{q}_1$.

Example: Let R_1 be the rotation about the x-axis by 90° and let R_2 be the rotation about the z-axis by 90°. Then

$$q_1 = \cos 45° + \sin 45°\, i = \frac{1}{\sqrt{2}}(1 + i)$$

$$q_2 = \cos 45° + \sin 45°\, k = \frac{1}{\sqrt{2}}(1 + k)$$

represent R_1 and R_2 respectively. Then $R_2 \circ R_1$ is represented by

$$\begin{aligned} q_2 q_1 &= \tfrac{1}{2}(1 + k)(1 + i) \\ &= \tfrac{1}{2}(1 + i + k + ki) \\ &= \tfrac{1}{2}(1 + i + j + k) \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{i + j + k}{\sqrt{3}}\right) \\ &= \cos 60° + \sin 60°\left(\frac{i + j + k}{\sqrt{3}}\right) \end{aligned}$$

which is, according to Theorem 2, a 120° rotation about the axis $i + j + k$.

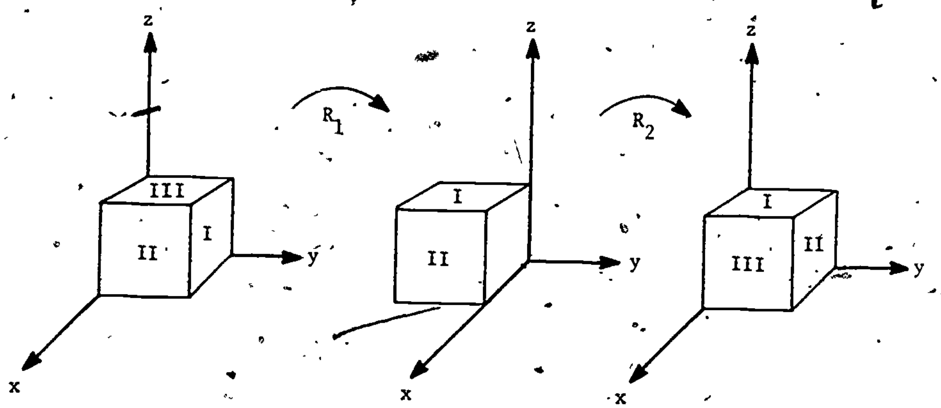


Figure 7. Composing rotations in R^3. If R_1 is a 90° rotation about i, and if R_2 is a 90° rotation about j, then the composition $R_2 \circ R_1$ is a 120° rotation about the vector $i + j + k$.

Exercise 6. In the example above, find the result of rotating first about k and then about i, rather than the reverse. Show that this result differs from the result in the example. Thus, composition of rotations is *not* commutative.

Exercise 7. Find the result of rotating first about i by angle 180° then about $i + j - k$ by 120°, and finally about j by 90°.

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11. THE ORIGINAL QUESTION ANSWERED

In the introduction we asked whether it were possible to find formulas that allow a straightforward way to find the result of composing two rotations whose axes and angles are given. It is certainly possible, and the general procedure, as you saw in the example of the previous section, and in the solutions of Exercises 6 and 7, is the following: Let

R_1 denote a rotation about n by angle 2θ,

and

R_2 denote a rotation about m by angle 2ϕ.

To determine the axis and angle of the rotation $R_2 \circ R_1$, we first take

$$q_1 = \cos\theta + \sin\theta\, n \quad \text{and} \quad q_2 = \cos\phi + \sin\phi\, m.$$

The norm of q_1 and q_2 are both equal to 1. To see this, let $n = n_2 i + n_3 j + n_4 k$. Then

$$\begin{aligned} |q_1| &= \sqrt{\cos^2\theta + \sin^2\theta(n_2^2 + n_3^2 + n_4^2)} \\ &= \sqrt{\cos^2\theta + \sin^2\theta(1)} \\ &= 1. \end{aligned}$$

The argument that $|q_2| = 1$ is similar, and we may conclude that

$$|q_2 q_1| = |q_2||q_1| = 1$$

as well. The polar form of $q_2 q_1$ is therefore

$$\begin{aligned} q_2 q_1 &= |q_2 q_1| (\cos \rho + \sin \rho\, u) \\ &= (1)(\cos \rho + \sin \rho\, u) \\ &= \cos \rho + \sin \rho\, u, \end{aligned}$$

where u is a unit vector in R^3. When we compare the polar form of $q_2 q_1$ with the expression

$$\begin{aligned} q_2 q_1 &= (\cos \phi + \sin \phi\, m)(\cos \theta + \sin \theta\, n) \\ &= \cos \phi \cos \theta + \sin \theta \cos \phi\, n + \sin \phi \cos \theta\, m + \sin \phi \sin \theta\, m n \\ &= (\cos \phi \cos \theta - \sin \phi \sin \theta\, m \cdot n) \\ &\quad + (\sin \theta \cos \phi\, n + \sin \phi \cos \theta\, m + \sin \phi \sin \theta\, m \times n), \end{aligned}$$

we see that

$$u = \frac{1}{r}(\sin \theta \cos \phi\, n + \sin \phi \cos \theta\, m + \sin \phi \sin \theta\, m \times n),$$

where

$$r = |\sin \theta \cos \phi\, n + \sin \phi \cos \theta\, m + \sin \phi \sin \theta\, m \times n|.$$

We also see that

$$\cos \rho = (\cos \phi \cos \theta - \sin \phi \sin \theta\, m \cdot n),$$

and that

$$\sin \rho = r.$$

These formulas for u and ρ give us the information that we seek about $R_2 \circ R_1$.

Exercise 8. Suppose an object rotates about some fixed point in such a way that a point on the object at $u = ai + bj + ck$ at time 0 is at $u(t) = R_t(u)$ at time t. Assume that R_t is a rotation about a *fixed* axis n by angle $\theta = \theta(t)$. Show that the velocity of the point $u(t)$ at time t is $u'(t) = \theta'(t)\, n \times u(t)$. (Note: As in calculus the velocity of a vector $u(t) = a(t)i + b(t)j + c(t)k$ is its term-by-term derivative. That is, $u'(t) = a'(t)i + b'(t)j + c'(t)k$, provided

the appropriate derivatives exist. Similarly, the derivative of a quaternion function of t is its term-by-term derivative.) It may be helpful to prove that the product rule holds for differentiating products of quaternions: $(pq)' = p'q + pq'$ with the corresponding rule holding for products of more than two quaternions. Incidentally, it is possible to generalize the formula for $u'(t)$ to the case where the axis of rotation n also depends on t, as it does in a wobbly spinning top.

Exercise 9: We could have introduced the multiplication operator by means of two other operators, the left and right multiplication maps. Let q be a *fixed* quaternion, and define R_q and L_q (for right and left multiplication) by

$$R_q(p) = pq$$

$$L_q(p) = qp.$$

Then $M_q = L_q \circ R_{\bar{q}}$. Interpret these two maps. (Hint: First write q in polar form as $r(\cos \theta + \sin \theta\, n)$ where r is the norm of q and n is a unit vector in R^3. Now calculate the effect of R_q and L_q on each of the basis elements $1, u, v, n$ where u, v, n form a right-hand orthonormal basis of R^3. Note that R_q and L_q are each linear.)

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12. HISTORICAL NOTE

The quaternions were invented by William Hamilton (1805-65), an English mathematician and mathematical physicist. Hamilton discovered the noncommutative rules for multiplication of i, j, and k in a flash of insight as he was walking over a bridge near Dublin. He is perhaps the first mathematician to consider the possibility of an algebraic system in which commutativity does not hold. His book *Elements of Quaternions* (London, 1866) was written at the time when linear algebra was in its birth and provides interesting reading. The quaternions were the first algebraic system to have been studied beyond the complex numbers. The modern notation for vectors was

introduced mainly through the efforts of the American physicist J.W. Gibbs (1839-1903). There were several emotionally bitter controversies between the early proponents of quaternionic algebra and those of vector algebra.

13. MODEL EXAMINATION

1. Find the following products:
   a. $i(i + j + k)$
   b. $(-1 + i)(2 - j)$
   c. $(i + j + k)^2$.

2. Find the multiplicative inverses of:
   a. 1
   b. $1 + i$
   c. $1 + i + j$.

3. Let $u = k - 2j$ and $v = i + 2j$. Calculate uv and vu. Verify the formulas in Exercise 2 (in the text) for dot and cross products of u and v.

4. Let $u = k - 2j$, $v = i + 2j$, and $w = 1 - k$. Show that

   $u(vw) = (uv)w \qquad u(v + w) = uv + uw$

   thus verifying associativity and distributivity in this special case.

5. Find the polar representations of
   a. i
   b. $i + j$
   c. $-i + j$
   d. $2 + i - j$.

6. Let $u = \cos\theta + \sin\theta\, n$ and $v = \cos\phi + \sin\phi\, n$, where n is a unit vector in R^3. Find uv and vu. Let R_1 and R_2 be the rotations associated with u and v, respectively. Verify that uv and vu both represent the rotation about n by angle $2(\theta+\phi)$, as they must from purely geometric considerations.

7. Let $q = 1 + i$. Find the polar representation of q. Let $p = xi + yj + zk$. Find $qp\bar{q}$. Show that $qp\bar{q}$ is p rotated about the x-axis by 90° followed by an expansion by factor 2.

8. Let R be the rotation about $i + j + k$ by angle 60° and S the rotation about $i - j + k$ by angle 90°. Find the axis and angle of rotation of R∘S.

9. Let R and S be the rotations given in the previous problem. Let T be the rotation about $i + k$ by angle 240°. Find the axis and angle of rotation of R∘S∘T.

14. ANSWERS TO EXERCISES

1. Now $n = x_1 i + x_2 j + x_3 k$. So

$$\begin{aligned} n^2 &= nn \\ &= x_1^2 i^2 + x_2^2 j^2 + x_3^2 k^2 \\ &\quad + x_1x_2(ij + ji) + x_1x_3(ik + ki) + x_2x_3(jk + kj) \\ &= -x_1^2 - x_2^2 - x_3^2 + 0 \\ &= -1, \end{aligned}$$

since n is a unit vector (where the second to the last equality follows from the rules for multiplication). Therefore, multiplication of any two elements of P yields

$$(a + bn)(c + dn) = ac + (bc + ad)n + bdn^2$$
$$= (ac - bd) + (bc + ad)n.$$

This is the formula one obtains in complex multiplication with $\sqrt{-1}$ replacing n. Similarly, addition of quaternions of the form $a + bn$ obeys the same rules as do complex numbers. The notation is convenient here: Denoting $\sqrt{-1}$ by i suggests how to embed the complex numbers in Q as the first two coordinates.

2. Calculate

$$uv = (ai + bj + ck)(xi + yj + zk)$$
$$= axi^2 + byj^2 = czk^2$$
$$+ ayij + bxji + azik + cxki + bzjk + cykj$$
$$= -(ax + by + cz) + (bz - cy)i + (cx - az)j + (ay - bx)k$$
$$= -(u \cdot v) + (u \times v).$$

By interchanging the roles of u and v we find

$$vu = -(v \cdot u) + (v \times u).$$

Now add and subtract to obtain formulas for the dot and cross product, noting that $u \cdot v = v \cdot u$ and $u \times v = -v \times u$.

3. Let $q = a + bi + cj + dk$; then

$$q\bar{q} = (a + bi + cj + dk)(a - bi - cj - dk)$$
$$= a^2 + b^2 + c^2 + d^2 = |q|^2$$

since all other terms cancel. Similarly for $\bar{q}q$. To show the third relation let $p = a + u$ where $u = bi + cj + dk$ and let $q = A + U$ where $U = Bi + Cj + Dk$. That is, a and A are the first coordinates of p and q. Now

$$\overline{pq} = \overline{(a + u)(A + U)}$$
$$= \overline{aA + Au + aU + uU}$$
$$= \overline{aA + Au + aU + [-(u \cdot U) + (u \times U)]}$$
$$= \overline{aA - (u \cdot U) + Au + aU + (u \times U)}$$
$$= aA - (u \cdot U) - Au - aU - (u \times U)$$

by the definition of conjugation and Exercise 2. Also

$$\bar{q} \cdot \bar{p} = (A - U)(a - u)$$
$$= Aa - aU - Au + Uu$$
$$= aA - Au - aU + [-(U \cdot u) + (U \times u)]$$
$$= aA - (u \cdot U) - Au - aU - (u \times U)$$

since $u \cdot U = U \times u$ and $(u \times U) = -(U \times u)$.

The last relation to be proved—that $|pq| = |p| \cdot |q|$—follows in much the same way. Assume that $p \neq 0$. (Otherwise the assertion is clear.)

$|p|^2|q|^2 = \bar{p}p\bar{q}q$	by the above
$= \bar{p}(pq)(\bar{q}\bar{p})r$	where r is the reciprocal to $\bar{p}$
$= \bar{p}(pq)\overline{(pq)}r$	by the above
$= \bar{p}|pq|^2 r$	
$= |pq|^2 \bar{p}r$	
$= |pq|^2$	

since r is the reciprocal to $\bar{p}$.

4. a. $$i + j + k = 0 + \sqrt{3}\left(\frac{i + j + k}{\sqrt{3}}\right)$$
$$= \sqrt{3}\left[\cos 90° + \sin 90°\left(\frac{i + j + k}{\sqrt{3}}\right)\right].$$

b. $$1 + i + j + k = 2\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{i + j + k}{\sqrt{3}}\right)\right]$$
$$= 2\left[\cos 60° + \sin 60°\left(\frac{i + j + k}{\sqrt{3}}\right)\right].$$

c. $$1 + k = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}k\right) = \sqrt{2}(\cos 45° + \sin 45° k).$$

d. $3 + 3j - 6k = \sqrt{54}\left(\frac{3}{\sqrt{54}} + \frac{3j - 6k}{\sqrt{54}}\right) = 3\sqrt{6}\left(\frac{1}{\sqrt{6}} + \sqrt{\frac{5}{6}}\cdot\frac{j - 2k}{\sqrt{5}}\right)$

$$\approx 3\sqrt{6}\left[\cos 66° + \sin 66°\left(\frac{j - 2k}{\sqrt{5}}\right)\right].$$

5. For ease of notation let $c = \cos\theta$ and $s = \sin\theta$. Then

$$\begin{aligned} M(n) &= (c + sn)n(c - sn) \\ &= (c + sn)(cn - sn^2) \\ &= (c + sn)(s + cn) \text{ by Problem 1} \\ &= cs + s^2n + c^2n + scn^2 \\ &= cs + n - sc \\ &= n, \end{aligned}$$

since $\cos^2\theta + \sin^2\theta = 1$. The argument that $M(v)$ has the required form is similar.

6. The quaternion that represents the rotation now is

$$\begin{aligned} &(\cos 45° + \sin 45°\, i)(\cos 45° + \sin 45°\, k) \\ &= \tfrac{1}{2}(1 + i)(1 + k) \\ &= \tfrac{1}{2}(1 + i - j + k) \\ &= \cos 60° + \sin 60°\left(\frac{i - j + k}{\sqrt{3}}\right). \end{aligned}$$

This represents a rotation about the $i - j + k$ axis by angle 120°.

7. The rotations are represented by

$$q_1 = \cos 90° + \sin 90°\, i = i$$

$$q_2 = \cos 60° + \sin 60°\left(\frac{i + j - k}{\sqrt{3}}\right) = \tfrac{1}{2}(1 + i + j - k),$$

and

$$q_3 = \cos 45° + \sin 45°\, j = \frac{1}{\sqrt{2}}(1 + j).$$

Thus the composition (note the order) is represented by

$$\begin{aligned} q_3q_2q_1 &= \frac{1}{2\sqrt{2}}(1 + j)(1 + i + j - k)i \\ &= \frac{1}{2\sqrt{2}}(1 + i + j - k + j - k - 1 - i)i \\ &= \frac{1}{2\sqrt{2}}(2j - 2k)i \\ &= \frac{1}{\sqrt{2}}(-k - j) \\ &= \cos 90° + \sin 90°\frac{(-j - k)}{\sqrt{2}}, \end{aligned}$$

which is a 180° rotation about the axis $-j - k$, or a 180° rotation about $j + k$.

8. First we note that if $p = a + bi + cj + dk$ and $q = A + Bi + Cj + Dk$ are each functions of t, then pq is the sum of terms such as $bCij = bCk$. Differentiating this term gives $b'Ck + bC'k$. Although the details are slightly tedious, it is at least easy to see that $(pq)'$ splits into two sets of terms one of which is the result of differentiating p, the other from differentiating q. And so the whole quaternion $(pq)'$ splits into $p'q + pq'$. This product rule extends to products of more than two quaternions just as in single variable calculus. Now $u(t) = R_t(u) = q(t)uq(t)$ where $q(t) = \cos[\tfrac{1}{2}\theta(t)] + \sin[\tfrac{1}{2}\theta(t)]n$. So the velocity of the point originally at u at time 0 at time t is

$$u'(t) = q'(t)u\overline{q(t)} + q(t)u\overline{q'(t)}$$

by the product rule. But $q'(t) = [\tfrac{1}{2}\theta'(t)](-\sin[\tfrac{1}{2}\theta(t)] + \cos[\tfrac{1}{2}\theta(t)]n)$. And similarly for $\overline{q'(t)}$. Notice that

$$\begin{aligned} q'(t) &= \tfrac{1}{2}\theta'(t)[-\sin(\tfrac{1}{2}\theta) + \cos(\tfrac{1}{2}\theta)n] \\ &= \tfrac{1}{2}\theta'n[\cos(\tfrac{1}{2}\theta) + \sin(\tfrac{1}{2}\theta)n] \\ &= \tfrac{1}{2}\theta'nq(t). \end{aligned}$$

Thus

$$\begin{aligned} u'(t) &= q'(t)u\overline{q(t)} + q(t)u\overline{q'(t)} \\ &= \tfrac{1}{2}\theta'[nq(t)u\overline{q(t)} + q(t)u\overline{q(t)}\overline{n}] \\ &= \tfrac{1}{2}\theta'[nu(t) - u(t)n] \\ &= \theta'[n \times u(t)] \end{aligned}$$

where the second to the last equality follows from the fact that $\overline{n} = -n$ for n in R^3 and the last equality follows from Exercise 2.

9. For ease of notation set $c = \cos\theta$ and $s = \sin\theta$. So $q = r(c + sn)$. That L_q and R_q are linear operators on $R^4 = Q$ is easy to see in the same way that M_q was seen to be linear. With u, v, n right-hand orthonormal, we calculate (see Figure 3):

$$R_q(1) = 1q = r(c + sn) = r(c1 + sn),$$

$$R_q(n) = nq = r(cn + sn^2) = r(-s1 + cn),$$

$$R_q(u) = uq = r(cu + sun) = r(cu - sv),$$

and

$$R_q(v) = vq = r(cv + svn) = r(su + cv).$$

This shows R_q to be a rotation in two planes (in the plane spanned by 1 and n by angle θ, in the u,v plane by angle $-\theta$), followed by an expansion by the real factor r. Similarly,

$$L_q(1) = r(c1 + sn),$$

$$L_q(n) = r(-s1 + cn),$$

$$L_q(u) = r(cu + sv),$$

and

$$L_q(v) = r(-su + cv).$$

This shows L_q to be a rotation in the $1,n$ plane by angle θ and in the u,v plane by angle θ, followed by an expansion by r.

15. ANSWERS TO THE MODEL EXAMINATION

1. a. $-1 - j + k$

   b. $-2 + 2i + j - k$

   c. -3

2. a. 1

   b. $2^{-1}(1 - i)$

   c. $3^{-1}(1 - i - j)$

3. $uv = 4 - 2i + j + 2k$, $\quad vu = 4 + 2i - j - 2k$,

   $u \cdot v = -4$, $\quad u \times v = -2i + j + 2k$

4. $u(vw) = 6 - 3i - j - 2k = (uv)w$, $\quad u(v + w) = 5 - j + 3k = uv + uw$

5. a. $\cos(90°) + \sin(90°)\, i$

   b. $\sqrt{2}\left[\cos(90°) + \sin(90°)\dfrac{i + j}{\sqrt{2}}\right]$

   c. $\sqrt{2}\left[\cos(90°) + \sin(90°)\dfrac{-i + j}{\sqrt{2}}\right]$

   d. $\sqrt{6}\left[\cos(35°) + \sin(35°)\dfrac{i - j}{\sqrt{2}}\right]$ approximately.

6. $uv = \cos(\theta + \phi) + \sin(\theta + \phi)\, n = vu$.

7. $q = \sqrt{2}[\cos(45°) + \sin(45°)\, i]$; $\quad qp\overline{q} = 2xi - 2zj + 2yk$

8. 121° rotation about $0.79i - 0.15j + 0.32k$ approximately.

9. 43° rotation about $0.51i + 0.58j - 0.64k$ approximately.

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STUDENT FORM 1

Request for Help

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________ Unit No. ______

Page ____ ○ Upper ○ Middle ○ Lower	OR	Section ____ Paragraph ____	OR	Model Exam Problem No. ____ Text Problem No. ____

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

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Name____________________ Unit No.__________ Date__________

Institution____________________ Course No.__________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain____________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain____________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)